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ON THE ASYMPTOTIC BEHAVIOR OF VELOCITY AND ON FORCES ACTING ON A BODY IN A STATIONARY STREAM OF VISCOUS FLUID

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It is shown that if the direction of the force acting on a body differs from that of the velocity of the oncoming stream, logarithmic factors appear in the asymptotic formula for velocity. A formula is derived for the force , which can be taken as the extension of the known Joukowsky theorem to the case of three-dimensional flow of viscous fluid past a body.

1. Statement of the problem. The flow of a stationary stream of viscous fluid past a body B of finite dimensions is considered. It is assumed that the boundary S of body B satisfies Liapunov's conditions. Let $u = (u_1, u_2, u_3)$ be the dimensionless velocity vector, p the dimensionless pressure, 2λ the Reynolds number, and $x = (x_1, x_2, x_3) \in R^3$ and $G = R^3 \setminus B$.

The flow of a viscous incompressible fluid is defined by the system of Navier-Stokes equations

$$\Delta u_j - 2\lambda \frac{\partial p}{\partial x_j} - 2\lambda u_k \frac{\partial u_j}{\partial x_k} = 0 \qquad (j = 1, 2, 3) \tag{1.1}$$

$$\partial u_k / \partial x_k = 0$$

The recurrent subscripts indicate summation from unity to three. The boundary conditions are

$$u_j|_S = 0, \quad \lim_{R\to\infty} u_j = u_{\infty j} = \delta_{1j} \tag{1.2}$$

where $R = |x|$ and δ_{ij} is the Kronecker delta.

It is shown in [1] that any solution of the boundary value problem (1.1), (1.2) (*) which has a finite Dirichlet integral (**)

$$J = \int_G |\nabla u|^2 \, dx \qquad (dx = dx_1 \, dx_2 \, dx_3) \tag{1.3}$$

satisfy condition

$$|u - u_\infty| = O(R^{-\alpha}) \qquad (R \to \infty)$$

for $\alpha > 1/2$, according to Finn, they are "physically acceptable" [8]. For physically acceptable solutions Finn obtained [8] the asymptotic formula

$$u_k(x) = u_{\infty k} + a_i H_{ik}(x) + O(R^{-3/2+\varepsilon}) \qquad (R \to \infty, \ k = 1, 2, 3) \tag{1.4}$$

where $a = 2\lambda F$, F is the dimensionless vector of force acting on the body B, ε is an arbitrarily small positive number, and $H(x)$ is the matrix of fundamental solutions of

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*) In [1] the boundary condition along the body are presented in a more general form.

**) Existence of such solutions is proved in [2 – 7].

the Oseen system of equations

$$\Delta H_{ij} - 2\lambda \frac{\partial q_i}{\partial x_j} - 2\lambda \frac{\partial H_{ij}}{\partial x_1} = \delta_{ij}\delta(x-y) \qquad (i, j = 1, 2, 3) \qquad (1.5)$$
$$\partial H_{ik} / \partial x_k = 0$$

A refinement of the asymptotic expansion (1.4) obtained by taking into account the nonlinear terms of the Navier-Stokes equations is presented in [9], where it is assumed that the direction of the vector of force F is the same as that of the oncoming stream u_∞. On this assumption

$$u_k(x) = u_{\infty k} + a_1 H_{1k}(x) + a_{ij}\frac{\partial H_{ki}}{\partial x_j}(x) + O[R^{-2+\varepsilon}(R - x_1 + 1)^{-1/2}] \quad (1.6)$$
$$(R \to \infty)$$

where a_{ij} are certain constants and $k = 1, 2, 3$.

A refinement of formula (1.4) is obtained below without the assumption about collinearity of vectors u_∞ and F. The formula for the force acting on the body B is also derived. In the latter formula the force is expressed in terms of the integral over a sphere of a fairly large radius whose center lies inside B, and an estimate is made of the residual term. This result represents an extension of the Joukowsky theorem to the case of stationary flow of a viscous fluid stream past a smooth body.

Extension of the Joukowsky theorem to the case of plane flow of a viscous fluid was considered in [10 – 12].

2. On the asymptotic behavior of velocity. In [9] the following formula

$$u_i(x) = u_{\infty i} + a_k H_{ik}(x) + a_{jk}\frac{\partial H_{ij}}{\partial x_k}(x) + I_{di}(x; v', v') +$$
$$O[R^{-2}(R - x_1 + 1)^{-1/2}\log^3 R]$$

where

$$I_{di}(x; v', v') = -2\lambda \int_G v_j' v_k' \frac{\partial H_{ij}}{\partial y_k} dy \qquad (i = 1, 2, 3), \qquad v' = aH \qquad (2.1)$$

was derived.

It can be shown (*) that, when $F \times u_\infty = 0$, then $I_i^d \ (x;\ v',\ v') = O[R^{-2+\varepsilon}(R - x_1 + 1)^{-1}]$ and, consequently, formula (1.6) is valid in this case. If, however, force F does not reduce to drag, the principal term of the asymptotic expansion of integral (2.1) $O(R^{-3/2}\ln R)$ is readily obtained by restricting computations to the remainder $O(R^{-3/2})$.

Let us now consider integral (2.1) for $F \times u_\infty \neq 0$. The fundamental solution of the Oseen system of equations (1.5) can be presented in the form

$$H_{ij} = \delta_{ij}\Delta\Phi - \frac{\partial^2\Phi}{\partial x_i \partial x_j} \qquad (i, j = 1, 2, 3)$$

$$2\lambda q_j = -\frac{\partial\Delta\Phi}{\partial x_j} + 2\lambda\frac{\partial^2\Phi}{\partial x_1 \partial x_j} = \frac{1}{4\pi}\frac{y_j - x_j}{|y - x|}$$

*) See K. I. Babenko and M. M. Vasil'ev, Asymptotic behavior of the solution of the problem of viscous fluid flow around a finite body. Preprint IPM AN SSSR, dep. № 4590-72, 1971.

$$\Phi = -\frac{1}{8\pi\lambda}\int_0^{\lambda s(x-y)} \frac{1-e^{-t}}{t}\,dt, \quad s(x-y) = |x-y| - x_1 + y_1$$

We have the following estimates:

$$|H_{ij}(x)| \leqslant CR^{-1}[s(x)+1]^{-1} \tag{2.2}$$
$$|\partial H_{ij} / \partial x_k| \leqslant CR^{-3/2}[s(x)+1]^{-3/2} \quad (k=1,2,3)$$

from which follows that

$$v_k' = -\frac{a_k e^{-\lambda s(x)}}{4\pi R} - \frac{1-\delta_{1k}}{R}\left[\frac{x_k\Phi''}{R}(a_2x_2+a_3x_3)+a_k\Phi'\right]+$$
$$O\{R^{-3/2}[s(x)+1]^{-1/2}\}$$

These formulas show that the integral (2.1) is the sum of integrals of the form

$$I(x) = \int_G [f(y)+g(y)]\,W(x-y)\,dy \tag{2.3}$$

where

$$f(y) = r^{-m} f_m[s(y)]\,[\omega_{kl}(\varphi)\sin^2\vartheta]^{2-m} \tag{2.4}$$
$$m=0,1,2; \quad k,l=1,2$$
$$|f_m(s)| < C(s+1)^{m-4} \tag{2.5}$$

$$|g(y)| \leqslant Cr^{-5/2}[s(y)+1]^{-3/2}, \quad \omega_{kl}(\varphi) = \cos^{\delta_{1k}+\delta_{1l}}\varphi\,\sin^{\delta_{2k}+\delta_{2l}}\varphi \tag{2.6}$$

where r,ϑ,φ are spherical coordinates and $W(x)$ is a continuously differentiable function for $x \neq 0$ which satisfies conditions

$$|W(x)| \leqslant CR^{-3/2}[s(x)+1]^{-3/2} \tag{2.7}$$

$$\left|\frac{\partial W}{\partial x_i}(x)\right| \leqslant CR^{-2-\delta_{1i}/2}[s(x)+1]^{-2+\delta_{1i}/2} \quad (i=1,2,3) \tag{2.8}$$

Let $B \subset d = \{y : |y| \leqslant 1\}$. We represent the region of integration G in the form of combined regions $d \cap G$ and $D_0 = R^3 \setminus d$. By virtue of estimate (2.2) for $R \to \infty$ the integral over region $d \cap G$ is $O(R^{-3/2})$. We decompose D_0 into regions

$$D_1 = \{y : 1 \leqslant |y| \leqslant R_0\} \quad (R_0 = hR,\ h = \text{const},\ 0 \leqslant h < 1/4)$$
$$D_2 = \{y : |y-x| \leqslant R_0\}, \quad D_3 = D_0 \setminus (D_1 \cup D_2)$$

and denote the integrals over these regions by I_1, I_2 and I_3, respectively.

Since $\forall y \in D_1$, $|x-y| \geqslant R - |y| \geqslant R(1-h)$, hence

$$\int_{D_1} g(y)\,W(x-y)\,dy \leqslant C_1R^{-3/2}\int_1^{R_0} r^{-1/2}dr\int_0^{\pi}[r(1-\cos\vartheta)+1]^{-3/2}\times$$
$$\sin\vartheta\,d\vartheta \leqslant CR^{-3/2}$$

and, consequently,

$$I_1 = \int_{D_1} f(y)\,W(x-y)\,dy + O(R^{-3/2})$$

Let us consider the integral

$$j = \int_{D_1} f(y) W(x-y)\,dy = W(x)\int_{D_1} f(y)\,dy + \int_{D_1} [W(x-y) - W(x)] f(y)\,dy$$

We introduce the notation

$$\Omega = \int_0^{2\pi} \omega_{kl}^{2-m}(\varphi)\,d\varphi$$

Using formula (2.4) for $m = 0, 1, 2$, we obtain

$$\int_{D_1} f(y)\,dy = 2^{2-m}\Omega \int_1^{R_0} \frac{dr}{r} \int_0^{2r} f_m(s) s^{2-m}\left[1 - (2-m)\frac{s}{2r} + \frac{(2-m)(1-m)}{2}\left(\frac{s}{2r}\right)^2\right] ds = 2^{2-m}\Omega \int_1^{R_0} \frac{dr}{r} \int_0^{2r} f_m(s) s^{2-m}\,ds + O(1) =$$

$$2^{2-m}\Omega \int_1^{R_0} \frac{dr}{r} \int_0^{\infty} f_m(s) s^{2-m}\,ds + O(1)$$

Thus

$$\int_{D_1} f(y)\,dy = A \ln R + O(1)$$

$$A = 2^{2-m}\Omega \int_0^{\infty} f_m(s) s^{2-m} ds \tag{2.9}$$

Let us now estimate the integral

$$j_1 = \int_{D_1} [W(x-y) - W(x)] f(y)\,dy = \int_{D_1} (y \cdot \mathrm{grad}\, W(\xi)) f(y)\,dy$$

$$\xi = x - \delta y \quad (0 < \delta < 1)$$

Since $|\xi| \geqslant |x| - \delta |y| \geqslant (1 - \delta h) R$, hence

$$|j_1| \leqslant CR^{-2} \int_{D_1} \left\{\frac{|y_1|}{R^{1/2}[s(\xi)+1]^{3/2}} + \frac{|y_2| + |y_3|}{[s(\xi)+1]^2}\right\} f(y)\,dy$$

Estimate

$$|f(y)| \leqslant C|y|^{-2}[s(y) + 1]^{-2}$$

is valid for the function $f(y)$. Using this estimate and passing to spherical coordinates, we obtain

$$|j_1| \leqslant C\left\{R^{-5/2}\int_1^{R_0} r\,dr\left[\int_0^{R^{-1/2}} \vartheta\,d\vartheta + \int_{R^{-1/2}}^{\pi} (r(1-\cos\vartheta)+1)^{-2}\sin\vartheta\,d\vartheta\right] + R^{-2}\int_1^{R_0} r\,dr\left[\int_0^{R^{-1/2}} \vartheta^2\,d\vartheta + \int_{R^{-1/2}}^{\pi} (r(1-\cos\vartheta)+1)^{-2}\sin^2\vartheta\,d\vartheta\right]\right\}$$

which implies that

$$|j_1| \leqslant CR^{-3/2}$$

Passing to the estimate of the integral I_2, we note that $\forall\, y \in D_2$, $|y| \geqslant CR$. Hence

$$|I_2| \leqslant C_1 R^{-2} \int_1^{R_0} r^{1/2} dr \int_0^{\pi} [r(1-\cos\vartheta)+1]^{-3/2} \sin\vartheta d\vartheta \leqslant CR^{-3/2}$$

For estimating I_3 we use the inequality $|x-y| \geqslant C|y|$, $\forall y \in D_3$.

$$|I_3| \leqslant C_1 \int_{R_0}^{\infty} r^{-3/2} dr \int_0^{\pi} [r(1-\cos\vartheta)+1]^{-2} \sin\vartheta d\vartheta \leqslant CR^{-3/2}$$

Combining the derived estimates, we obtain the following lemma.

Lemma. When conditions (2.4) – (2.8) are satisfied, for the integral (2.3) we have formula

$$I(x) = AW(x) \ln R + O(R^{-3/2}) \quad (R \to \infty)$$

where A is calculated by formula (2.9). Setting in formula (2.2)

$$f(y) + g(y) = -2\lambda v_j' v_k', \quad W = \partial H_{ij} / \partial y_k$$

we obtain on the basis of this lemma the following theorem.

Theorem 1. If surface S of body B satisfies Liapunov's conditions, then for solving the boundary value problem (1.1), (1.2) with a finite Dirichlet integral (1.3) we have formula

$$u_i(x) = u_{\infty i} + a_k H_{ik}(x) + l_{jk} \frac{\partial H_{ij}}{\partial x_k}(x) \ln R + O(R^{-3/2}) \quad (i = 1, 2, 3)$$

Coefficients l_{jk} are readily calculated with the use of formula (2.9).

3. The force acting on the body. The force acting on body B in a stationary stream of viscous fluid is determined in dimensionless form by formula

$$F_j = -\int_S \left[\frac{1}{2\lambda}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) - \delta_{jk} p\right] n_k d\sigma \quad (j = 1, 2, 3)$$

where n is the unit vector of the inward normal to surface S . The expression for this force can be represented according to [13] in the form

$$F_j = \int_\Sigma \left[\frac{1}{2\lambda}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) - \delta_{jk} p - u_j u_k\right] n_k d\sigma \tag{3.1}$$

where Σ is a sphere of a reasonably large radius R with its center lying inside B, and n is the unit vector of the outward normal to surface Σ. Using the velocity and pressure asymptotics and estimates of velocity derivatives it is possible to separate in formula (3.1) the principal terms and estimate the remainder.

It is shown in [9] that $|\partial u_j / \partial x_h| \leqslant CR^{-3/2} [s(x) + 1]^{-3/2+\varepsilon}$. Hence

$$\left| \int_\Sigma \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) n_k d\sigma \right| \leqslant CR^{-3/2} \int_0^{\pi} [R(1-\cos\vartheta)+1]^{-3/2+\varepsilon} R^2 \sin\vartheta d\vartheta \leqslant CR^{-1/2}$$

Let $v = u - u_\infty$. Then

$$F = -\int_\Sigma [pn + v(u_\infty \cdot n)] d\sigma + O(R^{-1/2}) \tag{3.2}$$

since

$$\int_\Sigma u_{\infty j} u_{\infty k} n_k d\sigma = 0, \quad \int_\Sigma u_{\infty j} v_k n_k d\sigma = 0$$

$$\left| \int_\Sigma v_j v_k n_k d\sigma \right| \leqslant C_1 \int_0^\pi [R(1-\cos\vartheta)+1]^{-2} \sin\vartheta d\vartheta \leqslant CR^{-1}$$

It follows from formula (3.2) that

$$F = -\int_\Sigma (p + u_\infty \cdot v) n d\sigma + u_\infty \times \Gamma + O(R^{-1/2}), \qquad \Gamma = \int_\Sigma (n \times v)\, d\sigma \quad (3.3)$$

The second term in formula (3.3) defines the component of force normal to the direction of the oncoming stream and is of a form similar to the lift in the Joukowsky theorem.

Let us transform the first term in (3.3)

$$X = -\int_\Sigma (p + v_1) n d\sigma$$

and show that it provides the principal term of drag. We use integral expressions for velocity and pressure (S_x is the boundary of region D_x)

$$v_1 = v_{s1} + V_1, \quad p = p_s + P$$

$$v_{s1} = -\int_S \left(\frac{\partial v_j}{\partial n} - 2\lambda p n_j\right) H_{1j} d\sigma - \int_S \left(\frac{\partial H_{1k}}{\partial y_1} + \frac{\partial H_{11}}{\partial y_k} + 2\lambda \delta_{1k} q_1\right) n_k d\sigma \quad (3.4)$$

$$p_s = -\int_S \left(\frac{\partial v_j}{\partial n} - 2\lambda p n_j\right) q_j d\sigma - \int_S \left(\frac{\partial q_1}{\partial n} - \lambda q_1 n_1\right) d\sigma \quad (3.5)$$

$$V_1 = -2\lambda \int_G v_j v_k \frac{\partial H_{1j}}{\partial y_k} dy \quad (3.6)$$

$$P = -2\lambda \int_{G \setminus D_x} v_j v_k \frac{\partial q_j}{\partial y_k} dy - \frac{1}{4\pi} \int_{S_x} \frac{v_k}{|x-y|} \frac{\partial v_j}{\partial y_k} n_j d\sigma + \quad (3.7)$$

$$\frac{1}{4\pi} \int_{D_x} \frac{1}{|x-y|} \frac{\partial}{\partial y_j} \left(v_k \frac{\partial v_j}{\partial y_k}\right) dy + 2 \int_{S_x} (q \cdot v)(v \cdot n)\, d\sigma$$

$$D_x = \{y : |y - x| \leqslant 1\}$$

Finn had shown [8] that estimates

$$|P(x)| \leqslant \begin{cases} CR^{-2} & \text{(uniformly)} \\ CR^{-2-2\beta} & \text{for } \vartheta \geqslant R^{-1/2+\gamma} \end{cases}$$

are valid for the quantity defined by (3.7), if $\beta < \gamma$ (ϑ is the angle between vector x and axis x_1). It follows from this that

$$\int_\Sigma P n d\sigma = O(R^{-1/2+\varepsilon})$$

where ε is an arbitrarily small positive number.

K. I. Babenko and the author of this paper had proved (*) the theorem which yields an estimate of the integral of the convolution type. Applying this theorem to the integral (3.6), we obtain

$$|V_1| \leqslant CR^{-3/2} \log^2 R\, [s(x) + 1]^{-1}$$

and, consequently,

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*) See footnote on page 71.

$$\left| \int_{\Sigma} V_1 n d\sigma \right| \leqslant C R^{-1/2+\varepsilon}$$

Thus

$$X = - \int_{\Sigma} (p_s + v_{s1}) n d\sigma + O(R^{-1/2+\varepsilon}) \tag{3.8}$$

Formulas (3.4) and (3.5) may be written as

$$v_{s1}(x) = - \int_{S} \left(\frac{\partial v_j}{\partial n} - 2\lambda p n_j \right) H_{1j} d\sigma + O\{R^{-2} [s(x) + 1]^{-1}\}$$

$$p_s(x) = - \int_{S} \left(\frac{\partial v_j}{\partial n} - 2\lambda p n_j \right) q_j d\sigma + O(R^{-3})$$

Substituting these expressions into formula (3.8) and using the asymptotic expansions

$$H_{1j} = \frac{x_j}{8\pi\lambda R^3} - (1 - \delta_{1j}) \frac{x_j e^{-\lambda s}}{8\pi R^2} - \delta_{1j} \frac{e^{-\lambda s}}{4\pi R} \left(1 + \lambda \frac{x_2 y_2 + x_3 y_3}{R} \right) + O\{R^{-2} [s(x) + 1]^{-1}\}$$

$$q_j = - \frac{x_j}{8\pi\lambda R^3} + O(R^{-3})$$

it is possible to show that $X_2 = O(R^{-1/2+\varepsilon})$, $X_3 = O(R^{-1/2+\varepsilon})$, and, consequently,

$$X = - e_1 \int_{\Sigma} (p_s + v_{s1}) n_1 d\sigma + O(R^{-1/2+\varepsilon})$$

where e_1 is the unit vector directed along the x_1-axis. The last formula shows that for $R \to \infty$ the direction of the principal term of vector X is the same as that of the oncoming stream velocity.

In this manner proof is given of the following theorem which is presented here in dimensional quantities.

Theorem 2. Let the flow past body B with boundary S satisfying Liapunov's conditions be determined by the solution of the boundary value problem (1.1), (1.2). The drag is then determined by formula

$$X = - e_1 \int_{\Sigma} (p + \rho u_\infty \cdot v) n_1 d\sigma + O(R^{-1/2+\varepsilon})$$

where ρ is the fluid density, and the sum of lift and side forces is determined by formula

$$Y = \rho u_\infty \times \Gamma + O(R^{-1/2+\varepsilon}) \tag{3.9}$$

Expressions for forces in [13] are given in terms of integrals over the control surface, however without rigorous proof and estimate of the remainder term. At the limit $R \to \infty$ formula (3.9) is the same as the corresponding formula in [13].

In concluding the author expresses his appreciation to K. I. Babenko for guidance in this work.

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ON THE HYPERSONIC FLOW PAST A LIFT AIRFOIL

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The hypersonic flow past a wing profile subjected to lift is considered. Effects of viscosity anf thermal conductivity in the region of flow outside the trail are neglected. An analogy is formulated which makes it possible to determine the velocity field by solving the problem of "directional" explosion in which not only energy but, also, momentum are imparted to gas. Motion within the viscous trail is specified by two terms of the asymptotic expansion of the solution of Navier-Stokes equations.

1. The outer region. Let us consider the hypersonic flow past a wing of infinite span. We denote the density of gas in the oncoming stream by ρ_∞ and by v_∞ its velocity in the direction of the x-axis of the Cartesian system of coordinates xy. We assume that upstream of the bow shock wave shown in Fig. 1 the pressure $p_\infty = 0$ and, consequently, the Mach number $M_\infty = \infty$. The gas is assumed to be perfect, i. e. to conform to the equation of state for such gas (the Clapeyron law) and that both specific